OTTAWA ONLINE
MAT 45143 – Introduction to Real Analysis

Course Description

Introduction to Real Analysis develops the theory of calculus carefully and rigorously from basic principles, giving the student of mathematics the ability to construct, analyze, and critique mathematical proofs in analysis.

This is a fully online, eight-week course. We will not meet face-to-face at any time.

Course Prerequisites

MAT-30243 Transition to Higher Mathematics

Required Text

Trench, W. (2003). Introduction to real analysis. Retrieved from http://ramanujan.math.trinity.edu/wtrench/texts/TRENCH_REAL_ANALYSIS.PDF (This book is available as a free e-textbook from the author. An electronic copy is available in the weekly materials for download. Release provided on page 2 of text.)

Course objectives:
After successful completion of the course, the student should be able to:

1. Identify the necessity of completeness for work in analysis.
2. Develop rigorously the concepts of convergence of real-valued sequences.
3. Develop rigorously the concepts and application of the Heine-Borel and Bolzano-Weierstrass theorems.
4. Identify, interpret and apply Cauchy's Convergence Criterion.
5. Rigorously discern and prove the value of a limit at a point, one-sided limits, limits at infinity, and limits of monotonic functions.
6. Establish the formal notion of continuity at a point, continuity over a set, and of uniform continuity of functions.
7. Identify and interpret types of discontinuities.
8. Establish the formal notion of derivative at a point.
9. Interpret, apply and prove the Mean Value Theorem.
10. Recognize and apply L'Hospital's rule in evaluating certain "indeterminate" limits.
11. Apply Taylor's Theorem to convex functions and to Newton's Method for the location of roots.
12. Formally define and prove properties concerning the Riemann-Stieltjes integral.
13. Formally define improper integrals and their convergence nature; utilize Dirichlet's test and change of variable.
14. Prove and utilize the first mean value theorem for integrals, the second mean value theorem for integrals, the fundamental theorem of calculus, change of variables, integration by parts.
15. Formally define a Lebesgue set of measure zero and use to find a necessary and sufficient condition for which a function on a close interval is Riemann-integrable.
16. Proof concepts associated with convergence and divergence to infinity.

17. Prove Cauchy’s convergence criteria for series of constants.
18. Determine nature of convergence of series of constants using the comparison test, integral test, ratio test and Raabe’s test.
19. Determine nature of absolute or conditional convergence, the effect of rearranging terms, and the multiplication of series.
20. Identify pointwise and uniform convergence of sequences and series of functions.
21. Utilize Cauchy’s uniform convergence criteria and Dirichlet’s test to prove uniform convergence of a series of functions.
22. Identify the criteria for using power series to integrate and differentiate.

Course Assignment Descriptions and Schedule At-A-Glance

You will have several opportunities to demonstrate your knowledge and understanding of the principles taught in this course. The primary means of evaluating your work will be through practical application of the material. In the event that you have difficulty completing any of the assignments for this course, please contact your instructor immediately. Please refer to the **Weekly Materials** section of the cyberclassroom for complete details regarding the activities and assignments for this course. The following is merely a summary.

Discussion contributions (160 points)

Initial Substantive Posts: Submit an initial response to each of the prompts provided each week by your instructor. Your initial post should be substantive (approximately ½ of a page in length) and must be posted by midnight, Central Time by Wednesday of each week. In your substantive post you are encouraged to use references (you may use your textbook); show evidence of critical thinking as it applies to the concepts or prompt and/or use examples of the application of the concepts to work and life. Proper punctuation, grammar and correct spelling are expected. Please use the **spell-check** function.

Required Replies: You must reply to at least two different peers per prompt. Your replies should build on the concept discussed, offer a question to consider, or add a differing perspective, etc. Rather than responding with, "Good post," explain why the post is "good" (why it is important, useful, insightful, etc.). Or, if you disagree, respectfully share your alternative perspective. Just saying "I agree" or "Good idea" is not sufficient for the posts you would like graded.

Posting Guidelines: Overall, postings must be submitted on at least two separate days of the week. It is strongly recommended you visit the discussion forum throughout the week to read and respond to your peers’ postings. You are encouraged to post more than the required number of replies.

(Please review the **Policies** section of Blackboard for further details.)

Week 1

Readings

Review the weekly lesson, videos and readings provided in the course.

- Section 1.1 The Real Number System
- Section 1.2 Mathematical Induction

Discussion

- Initial post to each prompt due by midnight, CT on Wednesday
- At least two replies to peers for each prompt due by midnight, CT on Sunday

Assignment(s)

Real Numbers

Provide answers to the following questions or problems. In writing your answers, be sure to mimic the method shown in the lesson.

1. Prove or give a counter example that $\sqrt{p}$ is irrational if p is prime. Prove or give a counterexample that the converse is true.
2. Problem 5 of Section 1.1 of the textbook.
3. Prove that for any nonempty set $S \subset \mathbb{R}$, $\inf(S) \le \sup(S)$ and give necessary and sufficient conditions for equality.
4. Show that any non-empty finite set $S \subset \mathbb{R}$ contains both its supremum and infimum. (Hint: use induction)
5. Problem 19 from Section 1.2 of the textbook.

Due: Sunday at Midnight, CT
Points Possible: 40

Week 2

Readings

- Review the weekly lesson, videos and readings provided in the course.
- Section 1.3 The Real Line
- Section 4.1 Infinite Sequences and Series

Discussion

- Initial post to each prompt due by midnight, CT on Wednesday
- At least two replies to peers for each prompt due by midnight, CT on Sunday

Assignment(s)

Sequence and Convergence

Provide answers to the following questions or problems. In writing your answers, be sure to mimic the method shown in the lesson.

1. Give an example of a bounded sequence that is not a Cauchy sequence.
2. Prove that a finite set has no cluster points.
3. Show that $\{x_n\}$ converges to L if and only if $\{x_n - L\}$ converges to 0.
4. Give an example of a set with exactly two cluster points.
5. Starting with $x_1 = 0.5$, find a solution in $[0,1]$ to $x^3 - 7x + 2 = 0$ accurate to 6 places. (That is, find a solution x_n such that $|x_{n+1} - x_n| < 0.0000001$.) If you know how to use one, use a spreadsheet to handle the computations.
6. Let $X = \{x_n\}$ be a sequence of strictly positive terms such that $\lim (x_{n+1}/x_n) = L > 1$. Show that X is unbounded and hence not convergent. (This result is used to establish the familiar Ratio Test one discovers in the calculus sequence.)
7. Find the limit of the sequence $x_n = (\sqrt{n} - 1)/(\sqrt{n} + 1)$. Prove your answer is correct.
8. A point $x \in \mathbb{R}$ is said to be an **interior point** of $A \subseteq \mathbb{R}$ in case there is a neighborhood S of x such that $S \subseteq A$. Show that $A \subseteq \mathbb{R}$ is open if and only if every point in A is an interior point of A.
9. A point $x \in \mathbb{R}$ is said to be a **boundary point** of $A \subseteq \mathbb{R}$ in case every neighborhood S of x contains points in A and points not in A. Show that for any set A, A and its complement, $\mathbb{R}$ - A, contain precisely the same boundary points. (Sets that contain their boundary points are called **closed sets**).
10. Find the limit of sequence defined recursively as $x_1 = 2$, $x_{n+1} = 1/(3 - x_n)$, $n \in \mathbb{N}$. You must first show the limit exists before attempting to find it.
11. Let $\{x_n\}$ be a Cauchy sequence such that x_n is an integer for all $n \in \mathbb{N}$. Show that $\{x_n\}$ is ultimately constant. (A sequence is **ultimately constant** if there exists a constant C and an $N_c \in \mathbb{N}$ for which $x_n = C$ for all $n \ge N_c$. For example, the sequence $\{7,4,8,9,12,17,3,2,1,1,1,1,1,1,\ldots\}$ is ultimately constant.)

Due: Sunday at Midnight, CT
Points Possible: 40

Week 3	
Readings	• Review the weekly lesson, videos and readings provided in the course. • Section 2.1 Functions and Limits • Section 2.2 Continuity • Section 4.2 Earlier Topics Revisited with Sequences
Discussion	• Initial post to each prompt due by midnight, CT on Wednesday • At least two replies to peers for each prompt due by midnight, CT on Sunday
Assignment(s)	**Limits of Functions** Provide answers to the following questions or problems. In writing your answers, be sure to mimic the method shown in the lesson. 1. Show that $\lim_{x \to 0} \sin(1/x^2)$ does not exist. 2. Prove that if $\lim_{x \to c} f(x)$ exists, then f is bounded on some neighborhood of c. (The function $f{:}D \to \mathbb{R}$ is bounded on some neighborhood of c if there exists a $\delta > 0$ and $M > 0$ such that $0 < \lvert x - c \rvert < \delta$, $x \in D$, implies $\lvert f(x) \rvert < M$.) 3. Give formal definition for a. $\lim_{x \to c^+} f(x) = \infty$. b. $\lim_{x \to c^-} f(x) = -\infty$. 4. Show that if the limit of a function at c exists then its value is unique. 5. Consider $f: (0,2) \to \mathbb{R}$ defined by $f(x) = x^x$. Assume f has limit at 0. Find it. (Hint: choose a sequence $\{x_n\}$ converging to zero such that the limit $\{f(x_n)\}$ is easy to determine.) Due: Sunday at Midnight, CT Points Possible: 40

Week 4	
Readings	• Review the weekly lesson, videos and readings provided in the course. • Section 2.2 Continuity
Discussion	• Initial post to each prompt due by midnight, CT on Wednesday • At least two replies to peers for each prompt due by midnight, CT on Sunday
Assignment(s)	**Continuity** Provide answers to the following questions or problems. In writing your answers, be sure to mimic the method shown in the lesson. 1.) Suppose $f: \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and $f(r) = 0$ for every $r \in \mathbb{Q}$. Prove that f is identically zero on $\mathbb{R}$. 2.) Problem #7 on page 70 of the text. 3.) Problem #10 on page 70 of the text. 4.) Problem #27 on page 72 of the text. 5.) Let $f: (0,1] \to \mathbb{R}$ be uniformly continuous on (0,1] and let $L := \lim \{f(1/n)\}$. Prove the if $\{x_n\}$ is any sequence in (0,1] such that $\lim \{x_n\} = 0$ then $\lim\{f(x_n)\} = L$. Deduce that if f is defined at 0 by $f(0) = L$, then f becomes continuous at 0. 6.) A function $f: \mathbb{R} \to \mathbb{R}$ is **periodic on $\mathbb{R}$ with period p** if there exists a $p > 0$ such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. Prove that a continuous periodic function on $\mathbb{R}$ is bounded and uniformly continuous on $\mathbb{R}$. Due: Sunday at Midnight, CT Points Possible: 40

Week 5	
Readings	• Review the weekly lesson, videos and readings provided in the course. • Section 2.3 Differentiable Functions of One Variable • Section 2.4 L'Hospital's Rule • Section 2.5 Taylor's Theorem
Discussion	• Initial post to each prompt due by midnight, CT on Wednesday • At least two replies to peers for each prompt due by midnight, CT on Sunday
Assignment(s)	**Differentiation** Provide answers to the following questions or problems. In writing your answers, be sure to mimic the method shown in the lesson. 1.) Prove "Fact 3" as found in these notes. 2.) Prove **Darboux's Theorem**: If f is differentiable on [a, b] and if k is a number between f'(a) and f'(b), then there is a least one point $c \in (a, b)$ such that f'(c) = k. 3.) Problem number 2 on page 85 of the text. 4.) Problem number 6 on page 85 of your text. 5.) Problem number 12 on page 86 of your text. 6.) Problem number 24 on page 87 of your text. 7.) Problem number 40 on page 97 of your text. 8.) Prove the statement given in Example 2 under the Taylor Theorem heading of these notes. Due: Sunday at Midnight, CT Points Possible: 40

Week 6	
Readings	• Review the weekly lesson, videos and readings provided in the course. • Section 3.1 Definition of the Integral • Section 3.2 Existence of the Integral • Section 3.3 Properties of the Integral • Section 3.4 Improper Integrals
Discussion	• Initial post to each prompt due by midnight, CT on Wednesday • At least two replies to peers for each prompt due by midnight, CT on Sunday
Assignment(s)	**Riemann Integration** Provide answers to the following questions or problems. In writing your answers, be sure to mimic the method shown in the lesson. 1. If $f: [a, b] \to \mathbb{R}$ is a bounded function such that f(x) = 0 except for $x \in \{c_1, c_2, \dots, c_n\} \subset [a, b]$, show that f is Riemann integrable on [a, b] and determine the Riemann integral value. 2. Suppose that f is any bounded function on [a, b] and that, for any number $c \in (a,b)$ the restriction of f to [c,b] is Riemann integrable. Show that f is integrable on [a, b] and that $$\int_a^b f = \lim_{c \to a^+} \int_c^b f .$$ 3. Exercise 14 on page 127 of the textbook. 4. Exercise 16 on page 127 of the textbook. 5. Prove that the function $f: [0,1] \to \mathbb{R}$ defined by f(0) =1, f(x) = 0 if x is irrational, and f(m/n) = 1/n, if $m, n \in \mathbb{N}$ are relatively prime is Riemann-integrable. 6. Prove that Dirichlet's discontinuous function is not Riemann-integrable. 7. State why each integral is improper, and determine whether or not it is convergent. If the improper integral is convergent, find its value. i. $\int_0^1 \ln(x)dx$; ii. $\int_1^2 \frac{1}{x \ln(x)} dx$; iii. $\int_1^2 \frac{x}{\sqrt{x-1}} dx$. Due: Sunday at Midnight, CT Points Possible:40

Week 7	
Readings	• Review the weekly lesson, videos and readings provided in the course. • Section 4.3 Infinite Series of Constants
Discussion	• Initial post to each prompt due by midnight, CT on Wednesday • At least two replies to peers for each prompt due by midnight, CT on Sunday
Assignment(s)	**Series** Provide answers to the following questions or problems. In writing your answers, be sure to mimic the method shown in the lesson. 1. Prove that absolute convergence implies convergence. 2. Use the comparison test to prove the limit comparison test. 3. Assume the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges to $\frac{\pi^2}{6}$. Use the integral test to "confirm" this result using S_{10} and $S_{20.}$ (Note: this exercise in no way proves the series has the suggested sum.) 4. Show that the series $$\frac{1}{1^2}+\frac{1}{2^3}+\frac{1}{3^2}+\frac{1}{4^3}+\cdots$$ is convergent, but that both the Ratio and the Root Tests fail to apply. 5. Let $\{a_n\}$ be a sequence of real numbers. Prove that $$\sum_{n=1}^{\infty}(a_n - a_{n+1})$$ converges if and only if $\{a_n\}$ converges. If the series $\sum(a_n - a_{n+1})$ converges, what is its sum? (The result is known as a **telescoping series**.) 6. Write an infinite series for the repeating decimal for the rational number 5/9 and prove your series converges to 5/9. 7. Suppose $\sum a_n$ converges absolutely and $\{b_n\}$ is bounded. Prove that $\sum a_n b_n$ converges absolutely. 8. If $\sum a_n$ converges absolutely, prove that $\sum a_n^2$ converges. Is the converse true? Is the statement true if $\sum a_n$ converges conditionally? 9. Determine the values of *r* for which $\sum nr^n$ converges. 10. Use Raabe's Test to show that if $\{a_n\}$ is a sequence of non-zero real numbers and if the limit $$a := \lim_{n\to\infty}\left[n\left(1-\left\|\frac{a_{n+1}}{a_n}\right\|\right)\right]$$ exists, then $\sum a_n$ is absolutely convergent when a > 1 and is not absolutely convergent when a < 1. Due: Sunday at Midnight, CT Points Possible: 40

Week 8	
Readings	• Review the weekly lesson, videos and readings provided in the course. • Section 4.4 Sequences of Series of Functions • Section 4.5 Power Series
Discussion	• Initial post to each prompt due by midnight, CT on Wednesday • At least two replies to peers for each prompt due by midnight, CT on **Saturday**
Assignment(s)	**Power Series** Provide answers to the following questions or problems. In writing your answers, be sure to mimic the method shown in the lesson. 1. Prove Weierstrass M-test. 2. Prove by induction that the function f(x) = exp(-1/x^2) for x ≠ 0 and f(0) = 0 has derivatives of all orders at every point in $\mathbb{R}$ and that all of these derivatives vanish at x = 0. Hence this function is not given by its Taylor expansion about x = 0. 3. Prove that if $\sum_{n=0}^{\infty} a_n x^n$ converges for x = x_0 and diverges for x = x_1, then a. $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $\lvert x\rvert < \lvert x_0\rvert$, and b. $\sum_{n=0}^{\infty} a_n x^n$ diverges for $\lvert x\rvert > \lvert x_1\rvert$. 4. How could one use the Ratio Test to establish criteria for the radius of convergence? 5. Suppose that $\sum a_n$ diverges and that {a_n} is bounded. Prove the radius of convergence of the power series $\sum a_n x^n$ is equal to 1. 6. Write the Taylor series for $\sqrt{1-x}$ centered at 0. Prove that the series converges to $\sqrt{1-x}$ for x ∈ (-1, 0]. Due: **Saturday**, midnight CT of Week 8 Points Possible: 40

* All online weeks run from Monday to Sunday, <u>except</u> the last week, which ends on Saturday.
** All assignments are due at midnight Central Time. (All submissions to the Blackboard system are date/time stamped in Central Time).

Assignments At-A-Glance

Assignment/Activity	Qty.	Points	Total Points
Weeks 1-8: Discussion*	-	20 per week	160
Week 1: Real Numbers	1	40	40
Week 2: Sequences and Convergence	1	40	40
Week 3: Limits of Functions	1	40	40
Week 4: Continuity	1	40	40
Week 5: Differentiation	1	40	40
Week 6: Riemann Integration	1	40	40
Week 7: Series	1	40	40
Week 8: Power Series	1	40	40
TOTAL POINTS			480

*Please refer to the **Policies** menu for more information about requirements for Discussions.

Grading Scale

Grade	Percentage	Points
A	90 to 100%	432 - 480
B	80 to 89%	384 - 431
C	70 to 79%	336 - 383
D	60 to 69%	288 - 335
F	< 60%	< 287

To access your scores, click on Grades in the Student Tools area in Blackboard.

Important Policies

All course-specific policies for this course are spelled out here in this syllabus. However, additional university policies are located in the Policies section of Blackboard. You are responsible for reading and understanding all of these policies. All of them are important. Failure to understand or abide by them could have negative consequences for your experience in this course.

Editorial Format for Written Papers

All written assignments are to follow the APA writing style guidelines for grammar, spelling, and punctuation. This online course includes information regarding the APA style under "Writing and Research Resources" in the **Resource Room** on the course menu in Blackboard.

Ottawa Online Late Policy

With instructor approval, assignments may be accepted for up to one week after the due date, but a minimum automatic deduction of 10% of the points will be assessed. The instructor also has the option of increasing this deduction percentage up to a maximum of 20%. Extenuating circumstances may be determined on rare occasions and an extension allowed without a deduction, but only at the sole discretion of the instructor.

Discussion board postings will not be accepted for credit when posted after the close of the discussion week. There are no exceptions to this rule; however, solely at the discretion of the instructor, the student may be allowed to submit an alternative assignment to make up for the points under extenuating circumstances. If granted, this should be an exception to the rule.

No assignments will be accepted after the last day of the course (end of term) unless arrangements have been made and "approved" by the instructor at least one week in advance.

Saving Work

It is recommended that you save all of your work from this course on your own computer or flash drive. The capstone course you take at the end of your program may require you to have access to this work for culminating assignments and/or reflections.

Academic Integrity

Plagiarism and cheating will not be tolerated at any level on any assignment. The reality of cyberspace has made academic dishonesty even more tempting for some, but be advised that technology can and will be used to help uncover those engaging in deception. If you ever have a question about the legitimacy of a source or a procedure you are considering using, ask your instructor. As the University Academic Council approved on May 29, 2003, *"The penalty for plagiarism or any other form of academic dishonesty will be failure in the course in which the academic dishonesty occurred. Students who commit academic dishonesty can be dismissed from the university by the provost/director*."

Please refer to **Academic Honesty** in the **Policies** section of the online course menu for important information about Ottawa University's policies regarding plagiarism and cheating, including examples and explanations of these issues.

Student Handbook

Please refer to your student handbook for all university regulations. The **Resource Room** on the course menu in Blackboard contains information about where to find the student handbook online for your campus.

Please see **Policies** in Blackboard for additional university policies.

Blackboard Technical Support

The Resource Room in Blackboard contains links to student tutorials for learning to use Blackboard as well as information about whom to contact for technical support. Ottawa University offers technical support from 8 a.m. to midnight Central Time for all students, staff, and faculty at no cost. See www.ottawa.edu/ouhelp for contact information.

Ottawa University Mission Statement

The mission of Ottawa University is to provide the highest quality liberal arts and professional education in a caring, Christ-centered community of grace, which integrates faith, learning and life. The University serves students of traditional age, adult learners and organizations through undergraduate and graduate programs.